

A GENERALIZED AXIS THEOREM FOR CUBE COMPLEXES

DANIEL J. WOODHOUSE

ABSTRACT. We consider a finitely generated virtually abelian group G acting properly and without inversions on a $\text{CAT}(0)$ cube complex X . We prove that G stabilizes a finite dimensional $\text{CAT}(0)$ subcomplex $Y \subseteq X$ that is isometrically embedded in the combinatorial metric. Moreover, we show that Y is a product of finitely many quasilines. The result represents a higher dimensional generalization of Haglund's axis theorem.

1. INTRODUCTION

A *CAT(0) cube complex* X is a cell complex that satisfies two properties: it is a geodesic metric space satisfying the $\text{CAT}(0)$ comparison triangle condition, and each n -cell is isometric to $[0, 1]^n$. We will call this metric the *CAT(0) metric* d_X and refer to [1] for a comprehensive account. A *hyperplane* $\Lambda \subseteq X$ is the subset of points equidistant between two adjacent vertices. Despite the brevity of this definition, hyperplanes are better understood via their combinatorial definition, and the reader is urged to consult the literature; see [6] [3] [7] for the required background. There also exists an alternative metric on the 0-cubes of X , that we will refer to as the *combinatorial metric* d_X^c , sometimes referred to as the ℓ^1 -metric. The combinatorial distance between two 0-cubes is the length of the shortest combinatorial path in X joining the 0-cubes. Equivalently, the combinatorial distance between two 0-cubes is the number of hyperplanes in X separating them. We will always assume that a group G acting on a $\text{CAT}(0)$ cube complex preserves its cell structure and maps cubes isometrically to cubes. A group G acts without *inversions* if the stabilizer of a hyperplane also stabilizes each complementary component. The requirement that the action be without inversions is not a serious restriction as G acts without inversions on the cubical subdivision.

A connected $\text{CAT}(0)$ cube complex X is a *quasiline* if it is quasiisometric to \mathbb{R} . The *rank* of a virtually abelian group commensurable to \mathbb{Z}^n is n . The goal of this paper will be the following theorem:

Theorem 4.3. *Let G be virtually \mathbb{Z}^n . Suppose G acts properly and without inversions on a $\text{CAT}(0)$ cube complex X . Then G stabilizes a finite dimensional subcomplex $Y \subseteq X$ that is isometrically embedded in the combinatorial metric, and*

$Y \cong \prod_{i=1}^m C_i$, where each C_i is a cubical quasiline and $m \geq n$. Moreover, $\text{Stab}_G(\Lambda)$ is a codimension-1 subgroup for each hyperplane Λ in Y .

Note that Y might not be a convex subcomplex, nor even isometrically embedded in the $\text{CAT}(0)$ metric.

Corollary 1.1. *Let A be a finitely generated virtually abelian group acting properly on a $\text{CAT}(0)$ cube complex X . Then A acts metrically properly on X .*

Let g be an isometry of X , and let $x \in X$. The *displacement of g at x* , denoted $\tau_x(g)$, is the distance $d_X(x, gx)$. The *translation length of g at x* , denoted $\tau(g)$, is $\inf\{\tau_x(g) \mid x \in X\}$. Similarly, if x is a 0-cube of X , we can define the *combinatorial displacement of g at x* , denoted $\tau_x^c(g)$, as $d_X^c(x, gx)$ and the *combinatorial translation length*, denoted $\tau^c(g)$, is $\inf\{\tau_x^c(g) \mid x \in X\}$. Note that τ_x, τ, τ_x^c , and τ^c are all conjugacy invariant. An isometry g of a $\text{CAT}(0)$ space is *semisimple* if $\tau_x(g) = \tau(g)$ for some $x \in X$, and G acts *semisimply* on a $\text{CAT}(0)$ space X if each $g \in G$ is semisimple.

If a virtually \mathbb{Z}^n group G acts metrically properly by semisimple isometries on a $\text{CAT}(0)$ space X , then the Flat Torus Theorem [1] provides a G -invariant, convex, flat $\mathbb{E}^n \subseteq X$. A virtually abelian subgroup is *highest* if it is not virtually contained in a higher rank abelian subgroup. If G is a highest virtually abelian subgroup of a group acting properly and cocompactly on a $\text{CAT}(0)$ cube complex X , then G cocompactly stabilizes a convex subcomplex Y which is a product of quasilinear, as above [8]. However, this theorem fails without the highest hypothesis. Moreover, most actions do not arise in the above fashion.

Despite the fact that the flat torus theorem will not hold under the hypotheses of Theorem 4.3, we can deduce the following:

Corollary 4.4. *Let G be virtually \mathbb{Z}^n . Suppose G acts properly and without inversions on a $\text{CAT}(0)$ cube complex X . Then G cocompactly stabilizes a subspace $F \subseteq X$ homeomorphic to \mathbb{R}^n such that for each hyperplane $\Lambda \subseteq X$, the intersection $\Lambda \cap F$ is either empty or homeomorphic to \mathbb{R}^{n-1} .*

The initial motivation for Theorem 4.3 and Corollary 4.4 was to resolve the following question posed by Wise. Although we have not found a combinatorial flat, Corollary 4.4 is perhaps better suited to applications (see [9]).

Problem 1.2. Let \mathbb{Z}^2 act freely on a $\text{CAT}(0)$ cube complex Y . Does there exist a \mathbb{Z}^2 -equivariant map $F \rightarrow Y$ where F is a square 2-complex homeomorphic to \mathbb{R}^2 , and such that no two hyperplanes of F map to the same hyperplane in Y ?

A *combinatorial geodesic axis* for g is a g -invariant, isometrically embedded, subcomplex $\gamma \subseteq X$ with $\gamma \cong \mathbb{R}$. Note that γ realizes the minimal combinatorial translation length of g . Theorem 4.3 is a high dimensional generalization of Haglund's combinatorial geodesic axis theorem. Haglund's proof involved an argument by contradiction, exploiting the geometry of hyperplanes. We reprove the result in Section 5 by using the dual cube complex construction of Sageev. The results are further support for Haglund's slogan "in CAT(0) cube complexes the combinatorial geometry is as nice as the CAT(0) geometry".

The following is an application of Theorem 4.3.

Corollary 1.3. *Let H be virtually \mathbb{Z}^n , and let $\phi : H \rightarrow H$ be an injection with $\phi \neq \phi^i$ for all $i > 1$. Then $G = \langle H, t \mid t^{-1}ht = \phi(h) : h \in H \rangle$ cannot act properly on a CAT(0) cube complex.*

Proof. Suppose that G acts properly on a CAT(0) cube complex X . After subdividing X we can assume that G acts without inversions. By hypothesis, there exists $a \in H$ such that $|\{\phi^i(a)\}| = \infty$. By Theorem 4.3 there is an H -equivariant isometrically embedded subcomplex $Y \subseteq X$ such that $Y \cong \prod_{i=1}^m C_i$ where each C_i is a cubical quasiline. As Y is a complete CAT(0) space, the Flat Torus Theorem [1] provides an H -invariant flat, $F \subseteq Y$, that is isometrically embedded in Y in the CAT(0) metric.

Fix a 0-cube y in Y . Choose a finite generating set S for H . Let Υ be the Cayley graph of H with respect to S , subdivided so that $d_\Upsilon(1, h) = \tau_y^c(h)$ for $h \in S$. Therefore Υ can be H -equivariantly mapped into Y . By the Švark-Milnor lemma, Υ with the combinatorial metric, and F with the CAT(0) metric are quasiisometric. Therefore $\tau_f(\phi^i(a)) \sim \tau_y^c(\phi^i(a))$ for $f \in F$. As $\tau_y^c(\phi^i(a)) = \tau_y^c(a)$ for all $i \in \mathbb{N}$, we deduce that $\{\tau_f(\phi^i(a))\}$ is a bounded set. This contradicts the properness of the action of H on F . \square

The above argument is inspired by the solvable subgroup theorem [1]. However, we have the following example of a solvable group which does act freely on a CAT(0) cube complex.

Example 1.4. Let $H = \langle a_1, a_2, \dots \mid [a_i, a_j] : i \neq j \rangle$. Note that H is the fundamental group of the nonpositively curved cube complex Y obtained from a 0-cube v , and 1-cubes $e_1, e_2, e_3 \dots$ with n -cubes inserted for every cardinality n collection of 1-cubes to create an n -torus. One should think of Y as an infinite cubical torus. The oriented loop e_i represents the element a_i .

Let $\phi : H \rightarrow H$ be the monomorphism such that $\phi(a_i) = a_{i+1}$. Let $G = H *_\phi = \langle t, a_1, a_2, \dots \mid [a_i, a_j] : i \neq j, t^{-1}a_it = a_{i+1} \rangle$ be the associated ascending HNN

extension. Note that G is generated by a_1 and t . There is a graph of spaces X obtained by letting Y be the vertex space and $Y \times [0, 1]$ be the edge space and identifying $(v, 1)$ and $(v, 0)$ with v , and the 1-cube $e_i \times \{1\}$ with e_i and $e_i \times \{0\}$ with e_{i+1} . Note that X is nonpositively curved, and therefore $G = \pi_1 X$ acts freely on the CAT(0) cube complex \tilde{X} , the universal cover of X .

Acknowledgements: I would like to thank Daniel T. Wise.

2. DUAL CUBE COMPLEXES

Let S be a set. A *wall* $\Lambda = \{\overleftarrow{\Lambda}, \overrightarrow{\Lambda}\}$ in S is a partition of S into two disjoint, nonempty subsets. The subsets $\overleftarrow{\Lambda}, \overrightarrow{\Lambda}$ are the *halfspaces* of Λ . A wall Λ *separates* $x, y \in S$ if they belong to distinct halfspaces of Λ . Let $K \subseteq S$. A wall Λ *intersects* K if K nontrivially intersects both $\overleftarrow{\Lambda}$ and $\overrightarrow{\Lambda}$. Let \mathcal{W} be a set of walls in S , then (S, \mathcal{W}) is a *wallspace* if for all $x, y \in S$, the number of walls separating x and y is finite. If Λ intersects K , then the *restriction of Λ to K* , is the wall in K determined by $\Lambda|_K = \{\overleftarrow{\Lambda} \cap K, \overrightarrow{\Lambda} \cap K\}$.

In this paper duplicate walls are not permitted in \mathcal{W} . Let \mathcal{H} be the set of halfspaces of corresponding to \mathcal{W} .

Example 2.1. Let X be a CAT(0) cube complex, and let $\Lambda \subseteq X$ be a hyperplane in X . The complement $X - \Lambda$ has two components, therefore defining a wall in X such that $\overleftarrow{\Lambda}$ is an open halfspace not containing Λ , and $\overrightarrow{\Lambda}$ is a closed halfspace containing Λ . Note that $\overleftarrow{\Lambda} \sqcup \overrightarrow{\Lambda} = X$. Let $L(\Lambda)$ and $R(\Lambda)$ denote the maximal subcomplexes contained in $\overleftarrow{\Lambda}$ and $\overrightarrow{\Lambda}$ respectively. Note that $L(\Lambda)$ and $R(\Lambda)$ are convex subcomplexes. Let \mathcal{W} be the set of walls determined by the hyperplanes in X . Then (X, \mathcal{W}) is the wallspace associated to X . Note that we are using Λ to denote both the hyperplane and the wall corresponding to the hyperplane.

A function $c : \mathcal{W} \rightarrow \mathcal{H}$ is a *0-cube* if $c[\Lambda] \in \{\overleftarrow{\Lambda}, \overrightarrow{\Lambda}\}$ and the following two conditions are satisfied:

- (1) For all $\Lambda_1, \Lambda_2 \in \mathcal{W}$ the intersection $c[\Lambda_1] \cap c[\Lambda_2]$ is nonempty.
- (2) For all $x \in S$, the set $\{\Lambda \in \mathcal{W} \mid x \notin c[\Lambda]\}$ is finite.

The *dual cube complex* $C(S, \mathcal{W})$ is the connected CAT(0) cube complex obtained by letting the union of all 0-cubes be the 0-skeleton. Two 0-cubes $c_1 \neq c_2$ are endpoints of a 1-cube if $c_1[\Lambda] = c_2[\Lambda]$ for all but precisely one $\Lambda \in \mathcal{W}$. An n -cube is then inserted wherever there is the 1-skeleton of an n -cube. The hyperplanes in $C(S, \mathcal{W})$ are identified naturally with the walls in \mathcal{W} . A proof of the fact that $C(S, \mathcal{W})$ is in fact a CAT(0) cube complex can be found in [5].

A point $x \in S$ determines a 0-cube c_x defined such that $x \in c_x[\Lambda]$ for all $\Lambda \in \mathcal{W}$. Condition (1) holds immediately since $x \in c_x[\Lambda]$ for all $\Lambda \in \mathcal{W}$. Condition (2) holds for c_x , since if $y \in S$ a wall Λ does not separate x and y , we can deduce that $y \in c_x[\Lambda]$, hence all but finitely many Λ satisfy $y \in c_x[\Lambda]$. Such 0-cubes are called the *canonical 0-cubes*.

Lemma 2.2. *Let X be a $CAT(0)$ cube complex. Let \mathcal{W} be a set of walls obtained from the hyperplanes in X . Let Z be a connected subcomplex of X , and let $\mathcal{W}_Z \subseteq \mathcal{W}$ be the subset of walls intersecting Z . Let \mathcal{V} be walls in \mathcal{W}_Z restricted to Z . Then (Z, \mathcal{V}) is a wallspace and $C(Z, \mathcal{V})$ embeds in $C(X, \mathcal{W})$ isometrically in the combinatorial metric.*

Proof. We first claim that the map $\mathcal{W}_Z \rightarrow \mathcal{V}$ is an injection. Suppose that $\Lambda_1, \Lambda_2 \in \mathcal{W}_Z$ are distinct walls. As Λ_1, Λ_2 intersects Z , and since Z is connected, there are 1-cubes e_1, e_2 in Z that are dual to the hyperplanes corresponding to Λ_1, Λ_2 . Therefore, both 0-cubes in e_1 belong in a single halfspace of $\Lambda_2|_Z$, so $\Lambda_1|_Z \neq \Lambda_2|_Z$.

We construct a map $\phi : C(Z, \mathcal{V}) \rightarrow C(X, \mathcal{W})$ on the 0-skeleton first. Let c be a 0-cube in $C(Z, \mathcal{V})$. We let $\phi(c) \in C(X, \mathcal{W})$ be the uniquely defined 0-cube such that $\phi(c)[\Lambda] \supseteq c[\Lambda|_Z]$ for $\Lambda|_Z \in \mathcal{V}$, and $\phi(c)[\Lambda] \supseteq Z$ for $\Lambda \in \mathcal{W} - \mathcal{W}_Z$. To verify that $\phi(c)$ is a 0-cube, first observe that $\phi(c)[\Lambda_1] \cap \phi(c)[\Lambda_2]$ is nonempty since $\Lambda_1|_Z \cap \Lambda_2|_Z \subseteq X$. Secondly, if $x \in X$ we need to show that $x \in \phi(c)[\Lambda]$ for all but finitely many $\Lambda \in \mathcal{W}$. Choose $z \in Z$, then $z \in c[\Lambda|_Z]$ for all $\Lambda|_Z \in \mathcal{V} - \{\Lambda_1|_Z, \dots, \Lambda_k|_Z\}$, hence $z \in \phi(c)[\Lambda]$ for all $\Lambda \in \mathcal{W}_Z - \{\Lambda_1, \dots, \Lambda_k\}$. Let $\{\Lambda_{k+1}, \dots, \Lambda_{k+\ell}\}$ be the set of walls in \mathcal{W} separating x and z . Then $x \in \phi(c)[\Lambda]$ for all $\Lambda \in \mathcal{W} - \{\Lambda_1, \dots, \Lambda_{k+\ell}\}$.

The 0-cubes are embedded since if $c_1 \neq c_2$, there exists $\Lambda|_Z \in \mathcal{V}$ such that $c_1[\Lambda|_Z] \neq c_2[\Lambda|_Z]$, hence $\phi(c_1)[\Lambda] \neq \phi(c_2)[\Lambda]$. If c_1, c_2 are adjacent 0-cubes in $C(Z, \mathcal{V})$, then $c_1[\Lambda|_Z] = c_2[\Lambda|_Z]$ for all $\Lambda|_Z \in \mathcal{V}$, with the exception of precisely one wall $\hat{\Lambda}|_Z$. Therefore, we can deduce that $\phi(c_1)[\Lambda] = \phi(c_2)[\Lambda]$ for all walls in \mathcal{W} , with the precise exception of $\hat{\Lambda}$. Therefore, the 1-skeleton of $C(Z, \mathcal{V})$ embeds in $C(X, \mathcal{W})$, which is sufficient for ϕ to extend to an embedding of the entire cube complex.

Consider $C(Z, \mathcal{V})$ as a subcomplex of $C(X, \mathcal{W})$. The set of hyperplanes in $C(Z, \mathcal{V})$ embeds into the set of hyperplanes in $C(X, \mathcal{W})$. To see that $C(Z, \mathcal{V})$ is an isometrically embedded subcomplex, let z_1, z_2 be 0-cubes in Z and γ be a geodesic combinatorial path in $C(Z, \mathcal{V})$ joining them. Each hyperplane dual to γ in $C(Z, \mathcal{V})$ intersects γ precisely once, and since the hyperplanes in $C(Z, \mathcal{V})$ inject to hyperplanes in $C(X, \mathcal{W})$, it is geodesic there as well. \square

Given a wall Λ associated to a hyperplane in X we let $N(\Lambda)$ denote the *carrier* of Λ , by which we mean the union of all cubes intersected by Λ .

Lemma 2.3. *Let S be a set and let \mathcal{W} be a set of walls of S . Let G be a group acting on (S, \mathcal{W}) . Let $\mathcal{V} \subseteq \mathcal{W}$ be a G -invariant subset. Then there is a G -equivariant function $\phi : C(S, \mathcal{W})^0 \rightarrow C(S, \mathcal{V})^0$. Moreover, $\phi^{-1}(z)$ is nonempty for all 0-cubes z in $C(S, \mathcal{V})$.*

Proof. Let c be a 0-cube in $C(S, \mathcal{W})$. Let $\phi(c)[\Lambda] = c[\Lambda]$ for $\Lambda \in \mathcal{V}$. It is immediate that ϕ is G -equivariant.

To verify $\phi(c)[\Lambda]$ is a 0-cube in $C(S, \mathcal{V})$ first note that $\phi(c_1)[\Lambda_1] \cap \phi(c_2)[\Lambda_2] \neq \emptyset$ for all $\Lambda_1, \Lambda_2 \in \mathcal{V}$, since $c_1[\Lambda_1] \cap c_2[\Lambda_2] \neq \emptyset$ for all $\Lambda_1, \Lambda_2 \in \mathcal{W}$. Secondly, for all $x \in S$ observe that $x \in \phi(c)[\Lambda]$ for all but finitely many $\Lambda \in \mathcal{V}$. Indeed, this is true for all but finitely many $\Lambda \in \mathcal{W}$.

To see that $\phi^{-1}(z)$ is non-empty for all 0-cubes z in $C(S, \mathcal{V})$ we determine a 0-cube x in $C(S, \mathcal{W})$ such that $\phi(x) = z$. Fix $s \in S$. Let $x[\Lambda] = z[\Lambda]$ for $\Lambda \in \mathcal{V}$. Suppose that $\Lambda \in \mathcal{W} - \mathcal{V}$. If $\overrightarrow{\Lambda} \supseteq z[\Lambda']$ for some $\Lambda' \in \mathcal{V}$ let $x[\Lambda] = \overrightarrow{\Lambda}$. Similarly if $\overleftarrow{\Lambda} \supseteq z[\Lambda']$. Otherwise, if Λ intersects $z[\Lambda']$ for all $\Lambda' \in \mathcal{V}$ then let $s \in x[\Lambda]$.

To verify that x is a 0-cube, consider the following cases to show $x[\Lambda_1] \cap x[\Lambda_2] \neq \emptyset$ for $\Lambda_1, \Lambda_2 \in \mathcal{W}$. If $\Lambda_1, \Lambda_2 \in \mathcal{V}$ then $x[\Lambda_1] \cap x[\Lambda_2] = z[\Lambda_1] \cap z[\Lambda_2] \neq \emptyset$. Suppose that $\Lambda_1 \in \mathcal{W} - \mathcal{V}$ and $x[\Lambda_1] \subseteq z[\Lambda'_1]$ for some $\Lambda'_1 \in \mathcal{V}$. If $\Lambda_2 \in \mathcal{V}$, then $x[\Lambda_1] \cap x[\Lambda_2] \supseteq z[\Lambda'_1] \cap z[\Lambda_2] \neq \emptyset$. If $\Lambda_2 \in \mathcal{W} - \mathcal{V}$ and $x[\Lambda_2] \subseteq z[\Lambda'_2]$ for some $\Lambda'_2 \in \mathcal{V}$ then $x[\Lambda_1] \cap x[\Lambda_2] \subseteq z[\Lambda'_1] \cap z[\Lambda'_2] \neq \emptyset$. If Λ_2 intersects $z[\Lambda]$ for all $\Lambda \in \mathcal{V}$, then $x[\Lambda_1] \cap x[\Lambda_2] \supseteq z[\Lambda'_1] \cap x[\Lambda_2] \neq \emptyset$. Finally if both $s \in x[\Lambda_1]$ and $s \in x[\Lambda_2]$, then their intersection will contain at least s .

Finally, we verify that for $s' \in S$ there are only finitely many $\Lambda \in \mathcal{W}$ such that $s' \notin x[\Lambda]$. Suppose, by way of contradiction, that there is an infinite subset of walls $\{\Lambda_1, \Lambda_2, \dots\} \subseteq \mathcal{W}$ such that $s' \notin x[\Lambda_i]$ for all $i \in \mathbb{N}$. We can assume, by excluding at most finitely many walls, that each $\Lambda_i \in \mathcal{W} - \mathcal{V}$. Similarly, by excluding finitely many walls, we can assume that Λ_i does not separate s and s' . Therefore, $s \notin x[\Lambda_i]$ for $i \in \mathbb{N}$. Therefore, by construction of x , there exist $\Lambda'_i \in \mathcal{V}$ such that $z[\Lambda'_i] \subseteq x[\Lambda_i]$, which implies that $s' \notin z[\Lambda'_i]$. There are infinitely many distinct Λ'_i , as otherwise there is a $\Lambda' \in \mathcal{V}$ such that $z[\Lambda'] \subseteq x[\Lambda_i]$ for infinitely many i , which would imply that infinitely many Λ_i separate s' from an element in the complement of $z[\Lambda']$. Therefore, infinitely many distinct walls $\Lambda'_i \in \mathcal{V}$ have $s' \notin z[\Lambda'_i]$, contradicting that z is a 0-cube in $C(S, \mathcal{V})$. \square

3. MINIMAL \mathbb{Z}^n -INVARIANT CONVEX SUBCOMPLEXES

The following Theorem is found in [2] (or less explicitly in [4]).

Theorem 3.1. *Let G be a finitely generated group that acts on a $CAT(0)$ cube complex X without a fixed point or inversions. Then there is a hyperplane in X that is stabilized by a codimension-1 subgroup of G .*

The goal of this section is to prove the following:

Lemma 3.2. *Let G be a finitely generated group acting without fixed point or inversions on a $CAT(0)$ cube complex X . There exists a minimal, G -invariant, convex subcomplex $X_o \subseteq X$ such that X_o contains only finitely many hyperplane orbits, and every X_o hyperplane stabilizer is a codimension-1 subgroup of G .*

Proof. Since G is finitely generated, by taking the convex hull of a G -orbit we obtain a G -invariant convex subcomplex $X_o \subseteq X$ containing finitely many G -orbits of hyperplanes. Assume that X_o is a minimal such subcomplex in terms of the number of hyperplane orbits.

Let (X, \mathcal{W}) be the wallspace obtained from the hyperplanes in X . Suppose that $\text{Stab}_G(\Lambda)$ is not a codimension-1 subgroup of G for some $\Lambda \in \mathcal{W}$. Let $G\Lambda \subseteq \mathcal{W}$ be the G -orbit of Λ . By Lemma 2.3 there is an G -invariant map $\phi : X_o^0 \rightarrow C(X, G\Lambda)^0$. Since $\text{Stab}_G(\Lambda)$ is not commensurable to a codimension-1 subgroup, Theorem 3.1 implies that there is a fixed 0-cube x in $C(X_o, G\Lambda)$. Lemma 2.3 then implies that $\phi^{-1}(x)$ is non-empty. Assuming that $\phi^{-1}(x) \subseteq \overleftarrow{\Lambda}$, then the intersection $\bigcap_{g \in G} gL(\Lambda)$ contains a proper, convex, G -invariant subcomplex of X_o , with one less hyperplane orbit. This contradicts the minimality of X_o . \square

The following Corollary follows since all codimension-1 subgroups of a rank n virtually abelian group are of rank $(n - 1)$.

Corollary 3.3. *Let G be a rank n , virtually abelian group acting and without fixed point or inversions on a $CAT(0)$ cube complex X . Then there exists a minimal, G -invariant, convex subcomplex $X_o \subseteq X$ such that X_o contains only finitely many hyperplane orbits, and every hyperplane stabilizer is a rank $(n - 1)$ subgroup of G .*

4. PROOF OF MAIN THEOREM

Definition 4.1. Regard \mathbb{R} as a $CAT(0)$ cube complex whose 0-skeleton is \mathbb{Z} . Let g be an isometry of X . A *geodesic combinatorial axis* for g is a subcomplex homeomorphic to \mathbb{R} that embeds isometrically in X .

Definition 4.2. Let (M, d) be a metric space. The subspaces $N_1, N_2 \subseteq M$ are *coarsely equivalent* if each lies in an r -neighbourhood of the other for some $r > 0$.

Theorem 4.3. *Let G be virtually \mathbb{Z}^n . Suppose G acts properly and without inversions on a $CAT(0)$ cube complex X . Then G stabilizes a finite dimensional*

subcomplex $Y \subseteq X$ that is isometrically embedded in the combinatorial metric, and $Y \cong \prod_{i=1}^m C_i$, where each C_i is a cubical quasiline and $m \geq n$. Moreover, $\text{Stab}_G(\Lambda)$ is a codimension-1 subgroup for each hyperplane Λ in Y .

Proof. By Corollary 3.3 there is a minimal, non-empty, convex subcomplex $X_o \subseteq X$ stabilized by G , containing finitely many hyperplane orbits, and $\text{Stabilizer}_G(\Lambda)$ is a rank $(n - 1)$ subgroup of G , for each hyperplane $\Lambda \subseteq X_o$.

Let $S = \{g_1, \dots, g_r\}$ be a generating set for G . Let $x \in X_o$ be a 0-cube. Let Υ be the Cayley graph of G with respect to S . Let $\phi : \Upsilon \rightarrow X_o$ be a G -equivariant map that sends vertices to vertices, and edges to combinatorial paths or vertices in X_o . Let $Q = \phi(\Upsilon)$. As G acts properly on X , and cocompactly on Υ , the graph Q is quasiisometric to G . Let \mathcal{W}_Q be the set of hyperplanes intersecting Q , and let (Q, \mathcal{W}_Q) be the associated wallspace. By Lemma 2.3 we know that $C(Q, \mathcal{W}_Q)$ is an isometrically embedded subcomplex of X_o . Fix a proper action of G on \mathbb{R}^n , and let $q : Q \rightarrow \mathbb{R}^n$ be a G -equivariant quasiisometry. Note that $\text{Stabilizer}_G(\Lambda)$ is a quasiisometrically embedded subgroup of G , for all $\Lambda \in \mathcal{W}_Q$. Thus $q(\Lambda \cap Q)$ is coarsely equivalent to a codimension-1 affine subspace $H \subseteq \mathbb{R}^n$. Moreover, $q(\overleftarrow{\Lambda} \cap Q)$ and $q(\overrightarrow{\Lambda} \cap Q)$ are coarsely equivalent to the halfspaces of H .

Let $n > 0$. Since there are finitely many orbits of hyperplanes in X_o , there are only finitely many commensurability classes of stabilizers. Therefore, we may partition \mathcal{W}_Q as the disjoint union $\bigsqcup_{i=1}^m \mathcal{W}_i$ where each \mathcal{W}_i contains all walls with commensurable stabilizers. For each $\Lambda_i \in \mathcal{W}_i$ let $q(\Lambda_i \cap Q)$ be coarsely equivalent to a codimension-1 affine subspace $H_i \subseteq \mathbb{R}^n$, stabilized by $\text{Stabilizer}_G(\Lambda_i)$. If $i \neq j$ then H_i and H_j are nonparallel affine subspaces, and therefore Λ_i and Λ_j will intersect in Q . Therefore, every wall in \mathcal{W}_i intersects every wall in \mathcal{W}_j if $i \neq j$, and thus $C(Q, \mathcal{W}_Q) \cong \prod_{i=1}^m C(Q, \mathcal{W}_i)$.

Finally, we show that $C(Q, \mathcal{W}_i)$ is a quasiline for each $1 \leq i \leq m$. As G permutes the factors in $\prod_{i=1}^m C(Q, \mathcal{W}_i)$, there is a finite index subgroup $G' \leq G$ that preserves each factor. For each i , the stabilizers $\text{Stab}_{G'}(\Lambda)$ are commensurable for all $\Lambda \in \mathcal{W}_i$. Therefore, there is a cyclic subgroup Z_i that is not virtually contained in any $\text{Stab}_{G'}(\Lambda)$ and thus acts freely on $C(Q, \mathcal{W}_i)$. As the stabilizers of $\Lambda \in \mathcal{W}_i$ are commensurable, all $q(\Lambda \cap Q)$ will be quasi-equivalent to parallel codimension-1 affine subspaces of \mathbb{R}^n , which implies that only finitely many Z_i -translates of Λ can pairwise intersect. As there are finitely many Z_i -orbits of Λ in \mathcal{W}_i , there is an upper bound on the number of pairwise intersecting hyperplanes in \mathcal{W}_i . Thus, there are finitely many Z_i -orbits of maximal cubes in $C(Q, \mathcal{W}_i)$, which implies that $C(Q, \mathcal{W}_i)$ is CAT(0) cube complex quasiisometric to \mathbb{R} . \square

We can now prove Corollary 4.4.

Corollary 4.4. *Let G be virtually \mathbb{Z}^n . Suppose G acts properly and without inversions on a $CAT(0)$ cube complex X . Then G cocompactly stabilizes a subspace $F \subseteq X$ homeomorphic to \mathbb{R}^n such that for each hyperplane $\Lambda \subseteq X$, the intersection $\Lambda \cap F$ is either empty or homeomorphic to \mathbb{R}^{n-1} .*

Proof. By Theorem 4.3 there is a G -equivariant, isometrically embedded, subcomplex $Y \subseteq X$, such that $Y = \prod_{i=1}^m C_i$, where each C_i is a quasiline, and $\text{Stab}_G(\Lambda)$ is a codimension-1 subgroup. Considering Y with the $CAT(0)$ metric, note that Y is a complete $CAT(0)$ metric space in its own right, and G acts semisimply on Y . By the Flat Torus Theorem [1] there is an isometrically embedded flat $F \subseteq Y$. Note that $F \subseteq X$ is not isometrically embedded. As $\text{Stab}_G(\Lambda)$ is a codimension-1 subgroup of G for each hyperplane Λ in X , the intersection $\Lambda \cap F = (\Lambda \cap Y) \cap F$ is either empty or, as $F \subseteq Y$ is isometrically embedded, the hyperplane intersection is an isometrically embedded copy of \mathbb{R}^{n-1} . \square

5. HAGLUND'S AXIS

The goal of this section is to reprove the following result of Haglund as a consequence of Corollary 4.4.

Theorem 5.1 (Haglund [3]). *Let G be a group acting on a $CAT(0)$ cube complex without inversions. Every element $g \in G$ either fixes a 0-cube of G , or stabilizes a combinatorial geodesic axis.*

Proof. As finite groups don't contain codimension-1 subgroups, Theorem 3.1 implies that if g is finite order then it fixes a 0-cube. Suppose that G does not fix a 0-cube, then $\langle g \rangle$ must act properly on X . By Corollary 4.4, there is a line $L \subset X$ stabilized by G , that intersects each hyperplane at most once at a single point in L . Let \mathcal{W}_L be the set of hyperplanes intersecting L . Note that the intersection points of the walls in \mathcal{W}_L with L is locally finite subset.

Fix a basepoint $p \in L$ that doesn't belong to a hyperplane intersecting L , and let x be the canonical 0-cube corresponding to p . Let $\Lambda_1, \dots, \Lambda_k$ be the set of hyperplanes separating p and gp , and assume that $p \in \overleftarrow{\Lambda}_i$. Reindex the hyperplanes such that $\overleftarrow{\Lambda}_1 \cap L \subseteq \overleftarrow{\Lambda}_2 \cap L \subseteq \dots \subseteq \overleftarrow{\Lambda}_k \cap L$. The ordering of the hyperplanes separating p and gp determines a combinatorial geodesic joining x and gx of length k , where the i -th edge is a 1-cube dual to Λ_i . This can be extended $\langle g \rangle$ -equivariantly, to obtain a combinatorial geodesic axis L_c , since each hyperplanes intersects L_c at most once. \square

REFERENCES

- [1] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999.
 - [2] V. N. Gerasimov. Semi-splittings of groups and actions on cubings. In *Algebra, geometry, analysis and mathematical physics (Russian) (Novosibirsk, 1996)*, pages 91–109, 190. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 1997.
 - [3] Frédéric Haglund. Isometries of CAT(0) cube complexes are semi-simple. pages 1–17, 2007.
 - [4] Graham A. Niblo and Martin A. Roller. Groups acting on cubes and Kazhdan’s property (T). *Proc. Amer. Math. Soc.*, 126(3):693–699, 1998.
 - [5] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)*, 71(3):585–617, 1995.
 - [6] Michah Sageev. Codimension-1 subgroups and splittings of groups. *J. Algebra*, 189(2):377–389, 1997.
 - [7] Daniel T. Wise. *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2012.
 - [8] Daniel T. Wise and J Woodhouse, Daniel. Classifying finite dimensional cubulations of tubular groups. pages 1–12, 2015. Submitted.
 - [9] Daniel J. Woodhouse. Classifying virtually special tubular groups. 2015. In Preparation.
- E-mail address:* `daniel.woodhouse@mail.mcgill.ca`